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VARIATIONAL THEORY OF DEFORMATIONS OF CURVED, TWISTED AND EXTENSIBLE ELASTIC RODS

by

Iradj G. Tadjbakhsh and Dimitris C. Lagoudas

ABSTRACT

The variational theory of three dimensional deformations of curved twisted and extensible elastic rods is obtained based entirely on the kinematical variables of position and rotations. The constitutive relations that define the resistive couples and the axial force as gradients of the strain energy function are established. A candidate for the strain energy function on the basis of classical assumptions is presented.

INTRODUCTION

In the Historical Introduction of the "A Treatise on the Mathematical Theory of Elasticity," Love (1892) narrates that in 1742 Daniel Bernoulli wrote to Euler suggesting that the differential equation of the elastica could be found by making the integral of the work done or the square of the curvature a minimum. Acting on this suggestion Euler was able to obtain the differential equation of the elastica and the various forms of it. Thus the concept of the strain energy was born and the foundations of the variational theory of elastic rods were laid out. The equilibrium equations that were much later developed by Love are applicable to an initially bent and twisted rod.

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Our aim in this paper is to establish a variational formulation for the title problem and in the process infer the existence of the strain energy function and determine the constitutive relations that relate this function to the bending and twisting couples and the axial force within the rod. This development together with equations of equilibrium and the geometry of deformation define a direct approach and an exact nonlinear theory for the three dimensional deformation of a one dimensional elastic medium capable of resisting bending twisting and extension. Going a step further, in order to actually construct an explicit form for the strain energy function, we enter the realm of hypothesis and use Kirchhoff's description of deformations in a thin rod. This view enables us to determine a strain energy function that can be used in engineering applications.

The recent history of investigations of the rod theories consists of developments along two separate streams, the direct approach and approximations from three–dimensional continuum. In the direct approach a one–dimensional continuum view is pursued and the medium is supposed endowed, in addition to its position, with vector fields, the directors, that are to be interpreted appropriately to define bending, twist and extension properties of a rod. This approach has its origins in the work of E. and F. Cosserat (1909) and numerous investigators have contributed to it among them Naghdi (1982), Naghdi and Rubin (1984), Whitman and DeSilva (1970), Green and Laws (1966), and Eriksen (1970). Extensive investigation into the qualitative aspects of the nonlinear theory such as questions of existence of solutions and global behavior have been carried out by Antman (1976). His basic work entitled "The Theory of Rods" (1972) describes these theories both as approximations to the three–dimensional continuum theory and as a one–dimensional continuum with directors. The work presented here, although pertains to a one–dimensional continuum, does not use directors, but is formulated entirely on the basis of kinematical quantities consisting of the position vector of points along the curve of centroids and the orientation angles of the cross sections of the rod relative to a fixed coordinate system. It is a generalization of the work of Tadjbakhsh (1966) in which the theory of planar motion of the extensible elastica was described.

The history of construction of approximate theories in the context of three–dimensional nonlinear continuum theory is also varied and to it many investigators including some of the above authors have contributed, see for example Green, Naghdi and Wenner (1974).

KINEMATICS

An elastica is a nearly uniform, slender rod of finite length. In the unstressed state the centroids of the cross section form a space curve C that is called the reference curve with an arc length s. The orientation of the principal axes of the cross sections vary continuously along the rod. This means that in the unstressed state the rod has arbitrary twist and curvatures. With respect to a fixed Cartesian frame the coordinates of points on the reference curve are denoted by $X_i(s)$, $i = 1, 2, 3$. The cross sectional area can be a slowly varying function of s and will be denoted by $A(s)$.

Attached to any point s of C a Cartesian coordinate system with axes y_i, $i=1,2,3$ will be assumed and will be referred to as the body axes. As shown in Fig. 1 the y_3 axis will be pointing in the direction of the increasing s and y_1 and y_2 axes will point along the principal axes of the cross section such that y_i – frame is a right handed system.

As the rod deforms the curve C acquires new configuration c. The arc length along c is denoted by $\xi(s)$ to account for extensibility of the rod. The position of a point s of C that in the unstressed state is $\mathbf{X}(s)$ at the stressed state will be $\mathbf{x}(s)$, both referred to the same fixed Cartesian frame. Noting that

$$\frac{\partial \xi}{\partial s} \equiv \xi' = (\mathbf{x}' \cdot \mathbf{x}')^{\frac{1}{2}}, \quad (' \equiv \frac{d}{ds}) \tag{1}$$

the strain e is defined by

$$e = \xi' - 1 \tag{2}$$

where $e>0$ denotes extension and $e<0$ contraction. The strict positivity of ξ' implies $-1 \leq e < \infty$.

Two sets of orthogonal unit vectors $\mathbf{e}_i$ and $\mathbf{a}_i$ will be assumed for the body frame y and the fixed frame x respectively. Denoting by $l_{ij}(s)$ the elements of the matrix of direction cosines between these two sets of unit vectors one has

$$\mathbf{a}_i = l_{ij}\mathbf{e}_j \tag{3}$$

$$e_i = l_{ji}\, n_j \tag{4}$$

Let us consider an element of the rod Ads, Fig. 1, and use the dynamical analogy of E.I. Routh. Love has noted that if the frame were to move with unit speed along the curve c such that at any point ξ of c it has the orientation of the y frame at that point then the angular velocities ω_1 and ω_2, about the y_1 and y_2 axes respectively, will be the principal curvatures κ_1 and κ_2 of the rod. Also the angular velocity ω_3 will be the twist curvature κ_3 of the rod. Thus the formulas that define angular velocities from direction cosines can be used to determine curvatures, provided time differentiation is replaced by differentiation with respect to ξ. The curvature vector is defined by

$$\boldsymbol{\kappa} = \kappa_i\, e_i \tag{5}$$

with

$$\kappa_i = \eta_{igh} \frac{d l_{jg}}{d\xi} l_{jh} \tag{6}$$

where $\eta_{ijk} = \epsilon_{ijk}(\epsilon_{ijk} + 1)/2$, (no sum on i, j, k), and ϵ_{ijk} is the alternator tensor with its non–zero components being $\epsilon_{123} = \epsilon_{231} = \epsilon_{321} = +1$ and $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$. Noting that $d/d\xi = (1 + e)^{-1} d/ds$, we define the three reference curvature parameters k_i by

$$k_i = (1+e)\kappa_i \tag{7}$$

For future use we record the formulas for derivatives of the unit vectors e_i

$$e_i' = \epsilon_{kji}\, k_j\, e_k \tag{8}$$

Associated with the unstrained state of the rod there will be the unit vectors E_i that bear the same relationship to the principal directions of the cross sections of the rod as do e_i in the strained state. In particular, if L_{ij} are the direction cosines of E_i with respect to n_i and K_i are the curvatures and twist of the rod in the unstrained state, then formulas (3) – (7) apply with l_{ij}, e_i and k_i replaced by L_{ij}, E_i and K_i, respectively, and e=0.

Since e_3 and $\partial x/\partial\xi$ are both unit tangents to the central line one has

$$x_i' = (1+e)l_{i3} \tag{9}$$

where x_i represent the components of $\mathbf{x}$ in the fixed frame i.e.

$$\mathbf{x} = x_i\, \mathbf{n}_i \tag{10}$$

Thus (9) is a differential constraint that relates the coordinates of the central line to direction cosines of the principal axes of the cross section of the rod.

The strain e and the three curvature parameters k_i form a set of four field variables that completely characterizes the deformation of the rod. In subsequent sections conjugate forces will be defined through the standard procedure of differentiating the strain energy function with respect to these kinematical variables.

EQUATIONS OF EQUILIBRIUM

Referring to the body set of axes $\mathbf{e}_i$ one can define the vector $\mathbf{F}$ of the resultant shear stresses F_1 and F_2 of the cross section and the axial stress resultant F_3. Similarly one may define the resultant couple stress vector $\mathbf{M}$ consisting of the bending moments M_1 and M_2 about the axes y_1 and y_2 and the torque M_3. Explicitly we have

$$\mathbf{F} = F_i\, \mathbf{e}_i \tag{11}$$

and

$$\mathbf{M} = M_i\, \mathbf{e}_i \tag{12}$$

The well known force and moment equilibrium equations are

$$\mathbf{F}' + \mathbf{f} = 0 \tag{13}$$

and

$$\mathbf{M}' + \mathbf{x}' \times \mathbf{F} + \mathbf{m} = 0 \tag{14}$$

where **f** and **m** represent the distributed force and couple acting on the rod, per unit undeformed length s.

The scalar components of equations of equilibrium can be expressed in the body set of axes. For this purpose one needs to express all vector quantities in terms of unit vectors e_i and use (8). Then (13) becomes

$$F_1' + k_2F_3 - k_3F_2 + f_1^y = 0 \tag{15a}$$

$$F_2' + k_3F_1 - k_1F_3 + f_2^y = 0 \tag{15b}$$

$$F_3' + k_1F_2 - k_2F_1 + f_3^y = 0 \tag{15c}$$

while (14) assumes the form

$$M_1' + k_2M_3 - k_3M_2 - (1 + e)F_2 + m_1^y = 0 \tag{16a}$$

$$M_2' + k_3M_1 - k_1M_3 + (1 + e)F_1 + m_2^y = 0 \tag{16b}$$

$$M_3' + k_1M_2 - k_2M_1 + m_3^y = 0 \tag{16c}$$

Here the superscript y on the components of **f** and **m** denote these components in the body reference frame.

To express the equations of equilibrium in the fixed frame we introduce the components of the stress resultants in that frame. Thus

$$F_i^x = l_{ij}F_j \ , \qquad M_i^x = l_{ij}M_j \tag{17}$$

Then (13) becomes

$$F_i^{x\prime} + f_i^x = 0 \tag{18}$$

and (14) assumes the form

$$M_1^{x\prime} - (1+e)(l_{i2} F_i^x) + m_1^x = 0 \tag{19a}$$

$$M_2^{x\prime} + (1+e)(l_{i1} F_i^x) + m_2^x = 0 \tag{19b}$$

$$M_3^{x\prime} + m_3^x = 0 \tag{19c}$$

CONSTITUTIVE RELATIONS

Either one of the set of equations (15)–(16) or (18)–(19) can be considered as the governing differential equations of equilibrium of the rod. These equations have to be supplemented with constitutive relations that define resultant axial stress F_3 and the resultant bending and twisting couples M_1, M_2, M_3 in terms of the axial strain e and curvatures k_1, k_2, k_3. For this purpose we assume the existence of a strain energy function $W(e, k_i)$ per unit undeformed length which is invariant under rigid body translations and rotations in the deformed configuration. We consider the variational problem of minimum of the total potential energy as the equivalent of equilibrium states of the rod. This potential can be expressed as

$$\begin{aligned} U[l_{ij}(\varphi_j), x_i, e] = \int_0^L \{ & W(e, k_i) + \lambda_i [x_i' - l_{i3}(1+e)] \\ & - f_i^x x_i - m_i^y \varphi_i\} ds - [\bar{M}_i \bar{\varphi}_i + \bar{F}_i \bar{x}_i]_0^L \end{aligned} \tag{20}$$

where $\bar{F}_i^0$, $\bar{M}_i^0$ and $\bar{F}_i^L$, $\bar{M}_i^L$ are the applied forces and moments at the end $s = 0$ and $s = L$ respectively.

The functions λ_i are Lagrange multipliers that allow the constraint (9) to be incorporated within the functional U. As a result x_i and l_{ij} can be regarded as independent variables. Additionally, the constraint (9) implies the definition (1)–(2) for the strain e and hence in (20) e can also be viewed as an independent variable. To see this we need to note that if each side of (9) is multiplied by itself we obtain $x_i'x_i' = (1+e)^2 l_{i3} l_{i3} = (1+e)^2$ which is a restatement of (1)–(2). The terms $f_i^x x_i$ and $m_i^y \varphi_i$ in the integrand of (20) represent the density of the potential of the applied forces and moments on the rod. The angles φ_i represent the rotations from E_i to e_i when these directions are assumed to issue from a common origin. These angles are determined through

$$\cos\varphi_i = e_i \cdot E_i = l_{ji} L_{ji}, \quad i=1,2,3 \quad \text{(sum only on j)} \tag{21}$$

The direction cosins l_{ji} are characterized by three orientation angles θ_1, θ_2, θ_3 that can be selected in a variety of ways and represent three finite rotations about the unit vectors e_i or n_i. If these rotations are properly selected any initial orientation of a cross section may be brought to any arbitrary final orientation. Kane et al. (1983) list at least 24 possibilities for order of rotations of the angles θ_1, θ_2, θ_3 about the body set of unit vectors e_i or the fixed set of unit vectors n_i. Thus in carrying out variation with respect to a particular φ_i (say φ_1) we need to select a particular sequence of rotation angles θ_i, the last of which coincides with φ_1. Same procedure must be used when carrying out variation with respect to φ_2 or φ_3.

With these preliminaries we note that the Euler equation corresponding to variations δx_i is simply $\lambda_i' + f_i^x = 0$, which when compared with (18) identifies λ_i with F_i^x. Next considering the variation with respect to e we obtain

$$\frac{\partial W}{\partial e} = F_i^x l_{i3} = F_3 \tag{22}$$

which is the constitutive relationship determining the axial force F_3 as the derivative of strain energy with respect to axial strain e.

We now turn to the Euler equation corresponding to the variation $\delta\varphi_1$. For orientation angles of the cross section we select the sequence of body rotations first $\theta_2 e_2$, second $\theta_3 e_3$ and third $\theta_1 e_1 = \varphi_1 e_1$. Then the matrix l of direction cosines is given by

$$l = B(\theta_2)C(\theta_3)A(\theta_1) = \begin{bmatrix} C_2C_3 & -C_1C_2S_3+S_1S_2 & S_1C_2S_3+C_1S_2 \\ S_3 & C_1C_3 & -S_1C_3 \\ -S_2C_3 & C_1S_2S_3+S_1C_2 & -S_1S_2S_3+C_1C_2 \end{bmatrix} \tag{23}$$

where

$$C_i \equiv \cos\theta_i \ , \quad S_i \equiv \sin\theta_i \tag{24}$$

$$A(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$C(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{25}$$

Subsequently we find from (7)

$$k_1 = \theta_2' S_3 + \theta_1' \tag{26a}$$

$$k_2 = \theta_2' C_1C_3 + \theta_3' S_1 \tag{26b}$$

$$k_3 = -\theta_2' S_1C_3 + \theta_3' C_1 \tag{26c}$$

From (20) we have

$$\{\frac{\partial W}{\partial k_i}\frac{\partial k_i}{\partial \varphi_1} - \frac{d}{ds}(\frac{\partial W}{\partial k_i}\frac{\partial k_i}{\partial \varphi_1'}) - (1+e)F_i^x[\frac{\partial l_{i1}}{\partial \varphi_1} - \frac{d}{ds}(\frac{\partial l_{i1}}{\partial \varphi_1'})]\} - m_1^x = 0 \tag{27}$$

Noting that $\partial/\partial\varphi_1 = \partial/\partial\theta_1$, we have from (26) $\partial k_1/\partial\theta_1 = 0$, $\partial k_2/\partial\theta_1 = k_3$, $\partial k_3/\partial\theta_1 = -k_2$ and from (23) $\partial l_{13}/\partial\varphi_1 = -l_{12}$. Using also the inverse of (17a), (27) becomes

$$\left(\frac{\partial W}{\partial k_1}\right)' + k_2\frac{\partial W}{\partial k_3} - k_3\frac{\partial W}{\partial k_2} - (1+e)F_2 + m_1^y = 0 \tag{28}$$

In exactly the same manner one can proceed to determine the Euler equation for a variation $\delta\varphi_2$. Now the consecutive sequence of body rotations $\theta_3 e_3$, $\theta_1 e_1$, $\theta_2 e_2$ is selected with $\theta_2 = \varphi_2$. Without going into details we obtain

$$\left(\frac{\partial W}{\partial k_2}\right)' + k_3\frac{\partial W}{\partial k_1} - k_1\frac{\partial W}{\partial k_3} + (1+e)F_1 + m_2^y = 0 \tag{29}$$

For variation of φ_3 we adopt the sequence of body rotations $\theta_1 e_1$, $\theta_2 e_2$, $\theta_3 e_3$ with $\theta_3 = \varphi_3$. The matrix of direction cosines is

$$l = A(\theta_1)B(\theta_2)C(\theta_3) = \begin{bmatrix} C_2C_3 & -C_2S_3 & S_2 \\ S_1S_2C_3 + C_1S_3 & -S_1S_2S_3 + C_1C_3 & -S_1C_2 \\ -C_1S_2C_3 + S_1S_3 & C_1S_2S_3 + S_1C_3 & C_1C_2 \end{bmatrix} \tag{30}$$

with curvatures given by

$$k_1 = \theta_1' C_2C_3 + \theta_2' S_3 \tag{31a}$$

$$k_2 = \theta_2' C_3 - \theta_1' C_2S_3 \tag{31b}$$

$$k_3 = \theta_3' + \theta_1' S_2 \tag{31c}$$

For this case l_{13} does not depend on θ_3 and hence the Euler variational equation assumes the form

$$\left(\frac{\partial W}{\partial k_3}\right)' + k_1\frac{\partial W}{\partial k_2} - k_2\frac{\partial W}{\partial k_1} + m_3^y = 0 \tag{32}$$

Comparison of equations (28), (29) and (32) with equations (16a) – (16c) respectively establishes the constitutive relations

$$M_i = \frac{\partial W}{\partial k_i} \quad i=1,2,3 \tag{33}$$

The specified boundary conditions must be consistent with

$$[(\hat{F}_i - F_i^x)\delta x_i + (\hat{M}_i - M_i)\delta\varphi_i]_0^L = 0 \tag{34}$$

which imply that at a boundary point either

$$F_i^x = \hat{F}_i \quad \text{or} \quad x_i = \bar{x}_i \tag{35}$$

and

$$M_i = \hat{M}_i \quad \text{or} \quad \varphi_i = \bar{\varphi}_i \tag{36}$$

where $\bar{x}_i$ and $\bar{\varphi}_i$ are specified position and orientation.

A STRAIN ENERGY FUNCTION

In order to gain an insight into the nature of the strain energy function we consider the strain of the lines and angles in the cross section of the rod. For this purpose we invoke the Kirchhoff hypothesis which assumes that the plane cross sections of rod that are normal to the axial direction in the unstrained state remain normal to the strained axial direction during deformation. Therefore the position vector to a material point in the cross–section before and after deformation can be given by

$$\mathbf{R} = \mathbf{X}(s) + y_1\mathbf{E}_1(s) + y_2\mathbf{E}_2(s) \tag{37}$$

and

$$\mathbf{r} = \mathbf{x}(s) + \alpha(s)[y_1\mathbf{e}_1(s) + y_2\mathbf{e}_2(s)] \tag{38}$$

respectively. The parameter $\alpha(s)$ is to be fixed by enforcing traction–free boundary conditions on the lateral surface of the rod.

Using the concept of extensional strains for stretching of line elements and distortion of angles between perpendicular lines as shear strains (Wempner, 1991), we define components of strain by

$$\epsilon_{ij} = \tfrac{1}{2}(\mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j) \tag{39}$$

where

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}}{\partial y_1} = \alpha \mathbf{e}_1, \qquad \mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial y_2} = \alpha \mathbf{e}_2,$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{r}}{\partial y_3} = \alpha' y_1 \mathbf{e}_1 + \alpha' y_2 \mathbf{e}_2 + \alpha y_1 \mathbf{e}_1' + \alpha y_2 \mathbf{e}_2' + (1+e)\mathbf{e}_3 \tag{40}$$

$$\mathbf{G}_1 = \frac{\partial \mathbf{R}}{\partial y_1} = \mathbf{E}_1, \; \mathbf{G}_2 = \frac{\partial \mathbf{R}}{\partial y_2} = \mathbf{E}_2, \; \mathbf{G}_3 = \frac{\partial \mathbf{R}}{\partial y_3} = y_1 \mathbf{E}_1' + y_2 \mathbf{E}_2' + \mathbf{E}_3 \tag{41}$$

Using (8) we can establish

$$\mathbf{e}_i' \cdot \mathbf{e}_j = \epsilon_{ijm} k_m \tag{42}$$

$$\mathbf{E}_i' \cdot \mathbf{E}_j = \epsilon_{ijm} K_m \tag{43}$$

where K_i is the curvatures and twist in the unstrained state. Therefore (9) yields as strain components

$$\epsilon_{11} = \epsilon_{22} = \tfrac{1}{2}(\alpha^2 - 1), \quad \epsilon_{12} = 0$$

$$\epsilon_{13} = \tfrac{1}{2}[\alpha(\alpha' y_1 - \alpha y_2 k_3) + y_2 K_3]$$

$$\epsilon_{23} = \tfrac{1}{2}[\alpha(\alpha' y_2 + \alpha y_2 k_3) - y_1 K_3]$$

$$\begin{aligned}\epsilon_{33} = {} & e + \tfrac{1}{2}e^2 - y_1[(1+e)\alpha k_2 - K_2] + y_2[(1+e)\alpha k_1 - K_1] - y_1 y_2(\alpha^2 k_1 k_2 - K_1 K_2) \\ & + \tfrac{1}{2}y_1^2[\alpha'^2 + \alpha^2(k_2^2 + k_3^2) - K_2^2 - K_3^2] + \tfrac{1}{2}y_2^2[\alpha'^2 + \alpha^2(k_1^2 + k_3^2) - K_1^2 - K_3^2]\end{aligned} \tag{44}$$

For a linear isotropic elastic material the non–zero stress components per unit strained area are

$$\sigma^{11} = (\lambda+G)\,(\alpha^2-1) + \lambda\epsilon^{33}\,, \quad \sigma^{12} = 0$$

$$\sigma^{22} = (\lambda+G)(\alpha^2-1) + \lambda\epsilon^{33}\,, \quad \sigma^{23} = 2G\epsilon_{23}$$

$$\sigma^{33} = \lambda(\alpha^2-1) + (\lambda+2G)\,\epsilon^{33}\,, \quad \sigma^{31} = 2G\epsilon_{31} \tag{45}$$

where λ and G are Lame's constant and the shear modulus, respectively. The traction per unit undeformed area is then given by

$$t^3 = \sigma^{3i}g_i \tag{46}$$

One can define the axial stress resultant F_3 by

$$\begin{aligned} F_3 = \int_A t^3\cdot e_3\, dA = \\ &= A(1+e)\left[(\lambda+2G)(e+\tfrac{1}{2}e^2) + \lambda(\alpha^2-1)\right] \\ &+ (\lambda+2G)\Big[I_1[(1+e)\alpha k_1-K_1]\alpha k_1 + I_2[(1+e)\alpha k_2-K_2]\alpha k_2 \\ &+ \tfrac{1}{2}I_1\,(1+e)(\alpha'^2 + \alpha^2k_1^2 - K_1^2 + \alpha^2k_3^2 - K_3^2) \\ &+ \tfrac{1}{2}I_2\,(1+e)(\alpha'^2 + \alpha^2k_2^2 - K_2^2 + \alpha^2k_3^2 - K_3^2)\Big] \end{aligned} \tag{47}$$

where $I_1 = \int_A y_2^2\, dA$ and $I_2 = \int_A y_1^2\, dA$. Similarly we have

$$\begin{aligned} M_1 = \int_A \alpha y_2 t^3\cdot e_3\, dA &= \alpha(\lambda+2G)\, I_1\left[(1+e)\alpha k_1 - K_1\right](1+e) \\ &+ I_1\alpha^2 k_1\left[\lambda(\alpha^2-1) + (\lambda+2G)(e+\tfrac{1}{2}e^2)\right] \end{aligned} \tag{48a}$$

$$\begin{aligned} M_2 = -\int_A \alpha\, y_1 t^3\cdot e_3 dA &= \alpha(\lambda+2G)\, I_2\left[(1+e)\alpha k_2 - K_2\right](1+e) \\ &+ I_2\alpha^2 k_2\left[\lambda(\alpha^2-1) + (\lambda+2G)(e+\tfrac{1}{2}e^2)\right] \end{aligned} \tag{48b}$$

$$M_3 = \int_A \alpha\,(y_1 t^3\cdot e_2 - y_2 t^3\cdot e_1)\,dA = JG\alpha^2(\alpha^2 k_3 - K_3) + J\alpha^2 k_3\left[\lambda(\alpha^2-1) + (\lambda+2G)(e+\tfrac{1}{2}e^2)\right] \tag{48c}$$

where $J=I_1+I_2$. One may note that the integrability conditions

$$\frac{\partial F_3}{\partial k_1}=\frac{\partial M_1}{\partial e},\ \frac{\partial F_3}{\partial k_2}=\frac{\partial M_2}{\partial e},\ \frac{\partial F_3}{\partial k_3}=\frac{\partial M_3}{\partial e},\ \frac{\partial M_1}{\partial k_2}=\frac{\partial M_2}{\partial k_1},\ \frac{\partial M_1}{\partial k_2}=\frac{\partial M_2}{\partial k_1},\ \frac{\partial M_2}{\partial k_3}=\frac{\partial M_3}{\partial k_2} \tag{49}$$

are satisfied. Hence existence of a strain energy function is assured and by integration we have

$$\begin{aligned} W = {} & A\,\frac{\lambda+2G}{2}\,e^2\left(1+e+\frac{e^2}{4}\right) - A\lambda\,(1-\alpha^2)(e+\tfrac{1}{2}e^2) \\ & + \lambda\, I_1\alpha^2\,(\alpha^2-1)\,\frac{k_1^2}{2} + (\lambda+2G)\, I_1\,\alpha\, k_1\left[\frac{1+2e^2+4e}{4}\,\alpha k_1 - (1+e)K_1\right] \\ & + \lambda\, I_2\alpha^2\,(\alpha^2-1)\,\frac{k_2^2}{2} + (\lambda+2G)\, I_2\,\alpha\, k_2\left[\frac{1+2e^2+4e}{4}\,\alpha k_2 - (1+e)K_2\right] \\ & + \frac{\lambda+2G}{4}\,(1+e)^2\left[I_1\,(\alpha'^2+\alpha^2 k_1^2 - K_1^2) + I_2\,(\alpha'^2+\alpha^2k_2^2-K_2^2)\right] \\ & + \lambda\, J\,\alpha^2\,(\alpha^2-1)\,\frac{k_3^2}{2} + \frac{\lambda+2G}{2}\,J\,(e+\tfrac{1}{2}e^2)\,(\alpha^2k_3^2 - K_3^2) + \frac{1}{2}\,G\,J\alpha\,(\alpha k_3 - K_3)^2 \end{aligned} \tag{50}$$

For an initially straight rod K_i should be set equal to zero in (47), (48) and (50). The above form of W reflects material isotropy, i.e., $W(e,k_1,k_2,k_3)=W(e,k_2,k_1,k_3)$, provided that $I_1=I_2$.

We note that positive curvatures imply positive bending moments and converseveiy negative curvatures imply negative bending moments provided that $e>(-1+1/\sqrt{3})$ for $\alpha=1$. This shows that equations (48) have a limited range of validity if the sense correspondence between moments and curvatures is be to retained. We also note the second order coupling between the squares of the curvatures and the axial strain e in (47), which implies that axial force can be generated by bending or twist only.

The parameter $\alpha(s)$ depends on the boundary conditions applied at the lateral surface of the rod. If the lateral surface is fixed, then $\alpha(s) = 1$. For zero tractions on the lateral surface, a condition appropriate for thin flexible rods is adopted according to which the average of σ^{11} and σ^{22} over the cross–section should vanish, i.e.

$$\int_A \sigma^{11} dA = \int_A \sigma^{22} dA = 0 \tag{51}$$

The above condition reduces to

$$H(\alpha, k_i, e) \equiv \alpha' J + \alpha^2 (I_1 k_1^2 + I_2 k_2^2 + J k_3^2 + \frac{A}{\nu})$$

$$- (I_1 K_1^2 + I_2 K_2^2 + J K_3^2) + 2Ae(1 + \tfrac{1}{2} e) - \frac{A}{\nu} = 0 \tag{52}$$

This equation should be interpreted as a differential equation for $\alpha(s)$ when k_i and e are known. To achieve this (52) is solved in an iterative procedure in which at every step k_i and e are known. We begin by writing (52) as $\lim_{n\to\infty} H(\alpha_{n+1}, k_i^n, e_n) = 0$ with n = 0, 1, 2, 3, ... For the first iteration (n = 0), $e_0 = 0$, $k_i^0 = K_i$, $\alpha_1 = 1$, and the six equations (15) – (16) after using (22), (33) and (50) contain only six unknown quantities F_1, F_2, e_1 and k_i^1 when the externally applied forces and moments are prescribed. Solution of this set of equations enables one to use k_i^1 in the curvature–orientation angle relations such as (31) to determine the latter i.e. θ_i. Now l_{ij} (θ_k) are known and one proceeds to determine φ_i from (21) and x_i from (9) using the appropriate boundary conditions. The solution for the first iteration is complete. One enters (52) with e_1 and k_i^1 and computes α_2 and the iteration proceeds.

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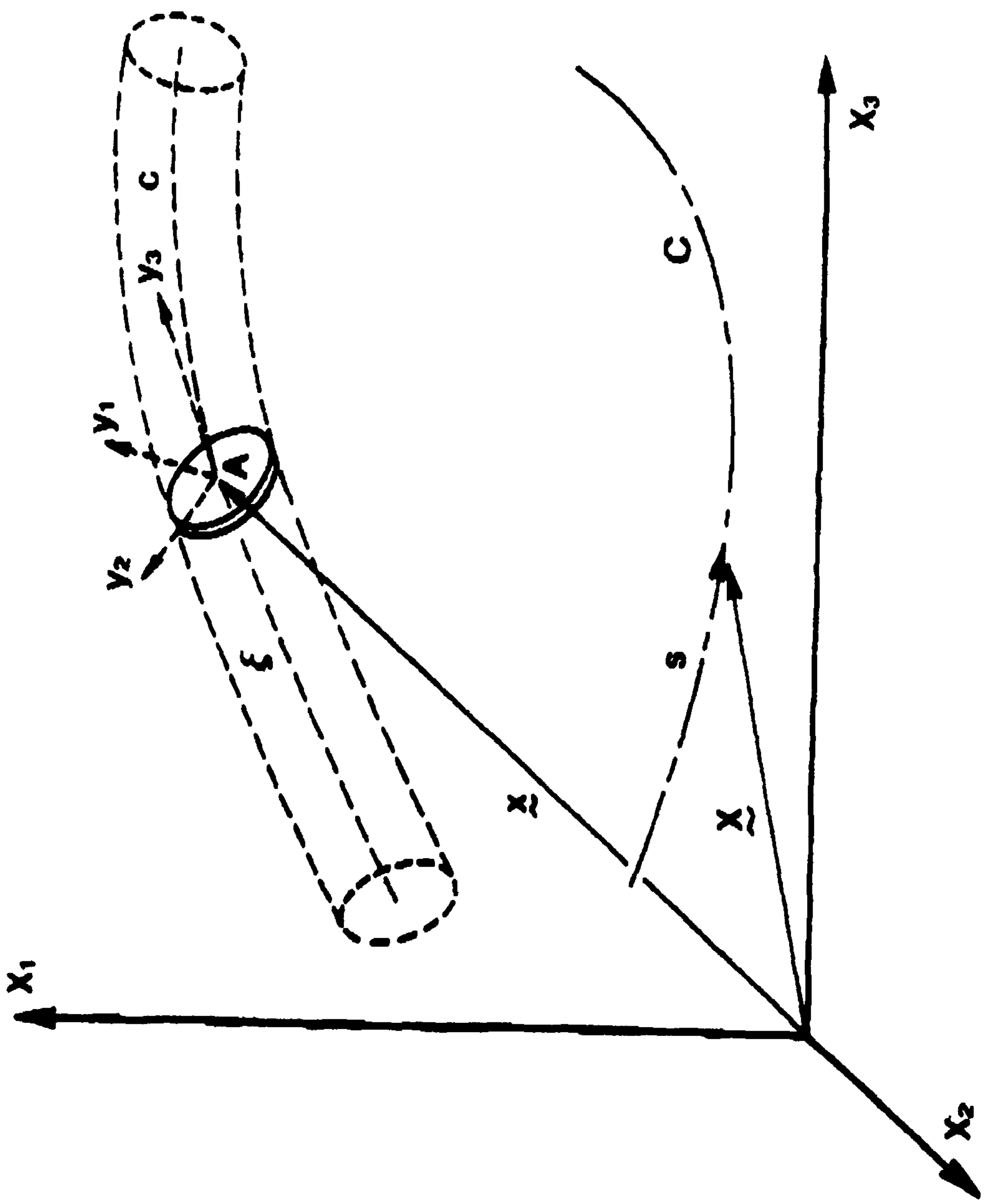

Fig. 1